Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 287 (2003) 473–486

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

First integrals and reduction of a class of nonlinear higher order ordinary differential equations

N. Euler [a] and P.G.L. Leach [b,*]

[a] *Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden*
[b] *School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, South Africa*

Received 15 January 2003

Submitted by R. O'Malley

Abstract

We propose a method for constructing first integrals of higher order ordinary differential equations. In particular third, fourth and fifth order equations of the form $x^{(n)} = h(x, x^{(n-1)})\dot{x}$ are considered. The relation of the proposed method to local and nonlocal symmetries are discussed.

Keywords: Nonlinear ordinary differential equations; First integrals; Integrability

1. Introduction

Ordinary differential equations arise in a multitude of contexts from the modelling of some phenomenon or process in which there is one independent variable or in the reduction of a partial differential equation for situations in which there is more than one independent variable. The result of the modelling process is an equation

$$E(t, x, \dot{x}, \ldots, x^{(n)}) = 0, \tag{1.1}$$

where, as always in this paper, an overdot denotes differentiation with respect to the independent variable t (equally a system of ordinary differential equations, but in this work we confine our attention to a single equation) and the next stage of the procedure, usually regarded as a separate study but in reality a vital component of modelling, is to solve the differential equation (1.1). The generic general ordinary differential equation does not have

* Corresponding author.
E-mail address: leach@math.aegean.gr (P.G.L. Leach).

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00544-4

a solution in closed form although, of course, the *existence* of a solution can be established under the standard conditions. Indeed it does not have an explicit solution in any form except for approximate solutions which may be computed numerically or by means of the standard perturbation methods and in the case of a chaotic system even this is of little value to describe the real evolution of the system. To make progress in the solution of (1.1) it is necessary to impose some form of constraint on the structure of the equation. Typically this constraint can be expressed as the existence of a Lie symmetry of the differential equation. To take a trivial example an autonomous equation obviously possesses the symmetry ∂_t, where t is the independent variable. Equally one could say that the constraint represented by the imposition of the symmetry ∂_t on the differential equation is that the equation is autonomous.

Examples of the class of equations treated in this paper are given below. Specifically we study the Rikitaki system (see Section 2.2), a system of the third order, and an equation of the fourth order which arises in the study of the generalised Emden–Fowler equation of index two (see Section 3.1).

Another way to impose a constraint on a differential equation is to require it to have some specific structure. One can be certain that there is symmetry implicit in the imposed structure, but it is not always as obvious as in the example above. For example of the Euler–Lagrange equation for the 'natural' Lagrangian, $\dot{x}^2/2 - V(x)$, videlicet

$$\ddot{x} + V'(x) = 0, \tag{1.2}$$

where the prime denotes differentiation with respect to the dependent variable x can be reduced to quadratures as

$$t - t_0 = \int \frac{dx}{(2E - V(x))^{1/2}} \tag{1.3}$$

and this is taken to indicate that the equation has been integrated. (We are not disagreeing with those who require the solution to be an analytic function, but accept the practicality of a lesser requirement.) That (1.2) is autonomous is sufficient to reduce the second order equation to a first order equation, but the possession of the single symmetry ∂_t does not explain the integrability of the original equation. For a generic potential (1.2) does not possess another Lie point symmetry. The source of the integrability of the first order equation is to be found in the nonlocal symmetry $\Gamma = (\int 1/\dot{x}^2\, dt)\partial t$ of (1.2) which becomes a point symmetry on the reduction of order induced by ∂_t [1,8]. The reduction of (1.2) to a quadrature was a consequence of the structure of the autonomous equation. In some manner, which a priori is not at all obvious, the constraint of the nonlocal symmetry causes it to be separable at the first order level and so reducible to a quadrature.

Our interest in this paper is the solution of nonlinear higher order equations through the calculation of their first integrals. Knowing well that we cannot solve explicitly a general higher order equation—after all we cannot do that with a first order equation—we address the question of the integrability of autonomous nonlinear higher order equations of the form

$$x^{(n)} = h(x, x^{(n-1)})\dot{x}, \tag{1.4}$$

in which the dependence of the equation upon $x^{(n)}$ and $\dot{x}$ is quite specific, that on x and $x^{(n-1)}$ is unspecified—apart from some modest requirement of good behaviour of the function h—and any other derivatives are excluded. Our main concern is with equations of the third, fourth and fifth orders. In subsequent sections we see how the assumed structure contributes to integrability in these specific cases. However, we firstly commence with a general proposal.

Basic proposition. *The differential equation* (1.4) *may always be written in the form*

$$\mathcal{D}^{(n-2)}\dot{x} = f(x, I_1), \tag{1.5}$$

where $\mathcal{D} = \dot{x}d/dx$ and I_1 is the first integral obtained by the solution of the first order equation

$$\frac{dx^{(n-1)}}{dx} = h(x, x^{(n-1)}). \tag{1.6}$$

In (1.4) we introduce the change of independent variable by writing $\dot{x}(x), \ddot{x}(x), \dots, x^{(n-1)}(x)$ so that the equation becomes

$$\frac{dx^{(n-1)}}{dx}\dot{x} = h(x, x^{(n-1)})\dot{x}. \tag{1.7}$$

On division by $\dot{x}$ (1.7) becomes (1.6). We assume that the solution of (1.6) can be written explicitly, videlicet

$$I_1 = g(x, x^{(n-1)}), \tag{1.8}$$

which is then inverted to give

$$x^{(n-1)} = f(x, I_1). \tag{1.9}$$

We may write (1.9) as an $(n-2)$th order equation by means of the change of independent variable from t to x which is the consequence of reduction of order using the obvious symmetry of (1.9), ∂_t. Recalling that we may rewrite the operator d/dt as the operator $\dot{x}d/dx$, (1.9) becomes

$$\mathcal{D}^{(n-2)}\dot{x} = f(x, I_1), \tag{1.10}$$

thereby establishing the proposition.

In the solution of (1.10) we obtain

$$\dot{x} = f(x, I_1, I_2, \dots, I_{n-1}), \tag{1.11}$$

in which $I_1, \dots, I_{n-1}$ are the constants of the integration of (1.10). Since these constants are the values of functions of $x^{(n-3)}, \dots, \ddot{x}$, x and I_1, they are first integrals of the original differential equation (1.4) as $x^{(n-1)}$ enters into the expression nontrivially through I_1. For the complete solution of (1.4) we must also have an invariant and this is the constant of integration in the quadrature

$$t - t_0 = \int \frac{dx}{f(x, I_1, I_2, \dots, I_{n-1})}, \tag{1.12}$$

i.e., the initial value of the independent variable t_0.

Remark. The application of our basic proposition to second order equations is clear and well known (consult, for example, [3, p. 137]). The general second order ordinary differential equation of the form of (1.4) is

$$\ddot{x} = h(x, \dot{x})\dot{x}, \tag{1.13}$$

which we rewrite as

$$\frac{d\dot{x}}{dx} = h(x, \dot{x}) \tag{1.14}$$

and, on integration of this first order equation, obtain

$$I_1 = g(x, \dot{x}). \tag{1.15}$$

We invert (1.15) to give

$$\dot{x} = f(x, I_1). \tag{1.16}$$

This form of (1.10) is a trivial differential equation for $\dot{x}(x)$. We immediately proceed to the final solution of the differential equation which is the performance of the quadrature

$$t - t_0 = \int \frac{dx}{f(x, I_1)}. \tag{1.17}$$

Here we aim to extend this method to equations of higher order.

2. The third order equation

2.1. The general equation

The third order form of (1.4) is

$$\dddot{x} = h(x, \ddot{x})\dot{x}, \tag{2.1}$$

a first integral for which is

$$I_1 = g(x, \ddot{x}) \tag{2.2}$$

obtained by the integration of the first order differential equation

$$\frac{d\ddot{x}}{dx} = h(x, \ddot{x}). \tag{2.3}$$

We invert (2.2) for $\ddot{x}$ and use the Basic proposition to express it as

$$\dot{x}\frac{d\dot{x}}{dx} = f(x, I_1) \quad \Leftrightarrow \quad \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = f(x, I_1) \tag{2.4}$$

from the formal integration of which we obtain the first integral

$$I_2 = \frac{1}{2}\dot{x}^2 - \int f(x, I_1)\, dx \tag{2.5}$$

and the solution of Eq. (2.1) is reduced to the quadrature

$$t - t_0 = \int \frac{dx}{(2I_2 + \int f(x, I_1)\, dx)^{1/2}}, \tag{2.6}$$

where t_0 is the invariant required to introduce the independent variable t into the solution.

A first integral/invariant $I(t, x, \ldots, x^{(n-1)})$, associated with a point (contact) symmetry Γ of an nth order scalar ordinary differential equation, $E(t, x, \ldots, x^{(n)}) = 0$, satisfies the dual requirements that

$$\Gamma^{(n-1)} I = 0 \quad \text{and} \quad \left.\frac{dI}{dt}\right|_{E=0} = 0. \tag{2.7}$$

When both of (2.7) are regarded as first order linear partial differential equations, the $n+1$ variables of the former reduce to n characteristics which become the n variables of the latter thereby reducing to $n-1$ characteristics, each of which is a first integral of the original differential equation. Thus in the case of our third order ordinary differential equation there are two first integrals associated with the obvious Lie point symmetry ∂_t. To each first integral there must be associated at least three symmetries [2]. In terms of the argument using characteristics each first integral is precisely specified, up to an arbitrary function of itself, by

$$\Gamma_1^{[2]} I = 0, \qquad \Gamma_2^{[2]} I = 0, \qquad \left.\frac{dI}{dt}\right|_{E=0} = 0, \tag{2.8}$$

where, obviously, $\Gamma_1 = \partial_t$. In view of the structure of (2.1) one should look to symmetries of the form $\xi\partial_t$, of which (2.1) has three, as solutions of the linear third order ordinary differential equation

$$\dddot{\xi}\,\dot{x} + 3\ddot{\xi}\ddot{x} + 3\dot{\xi}\,\dddot{x} = \dot{\xi}\dot{x}h + (2\dot{\xi}\ddot{x} + \ddot{\xi}\dot{x})\frac{\partial h}{\partial \ddot{x}}\dot{x}, \tag{2.9}$$

including the obvious one flowing from $\dot{\xi} = 0$, videlicet ∂_t.

When (2.1) is invoked, (2.9) becomes

$$\begin{aligned} &\dddot{\xi}\,\dot{x} + 3\ddot{\xi}\ddot{x} + 2\dot{\xi}\,\dddot{x} = (2\dot{\xi}\ddot{x} + \ddot{\xi}\dot{x})\frac{\partial h}{\partial \ddot{x}}\dot{x} \\ &\quad \Leftrightarrow \quad (\ddot{\xi}\dot{x} + 2\dot{\xi}\ddot{x})^{\cdot} = (\ddot{\xi}\dot{x} + 2\dot{\xi}\ddot{x})\frac{\partial h}{\partial \ddot{x}}\dot{x}, \end{aligned} \tag{2.10}$$

which is formally integrated to give

$$\xi = A + B\int \frac{dx}{\dot{x}^3} + C\int \frac{1}{\dot{x}^3}\left\{\exp\left[\int \frac{\partial h}{\partial \ddot{x}}\, dx\right] dx\right\} dx, \tag{2.11}$$

the degree of formality being reduced by use of the inversion of (2.2) to replace $\ddot{x}$ by $f(x, I_1)$. The relation $\dot{x}\, dt = dx$ has been used to obtain the integrands on the right of (2.11). We have the three symmetries

$$\begin{aligned} &\Gamma_1 = \partial_t, \qquad \Gamma_1 = \left(\int \frac{dx}{\dot{x}^3}\right)\partial_t = F_2(x, I_1, I_2)\partial_t, \\ &\Gamma_3 = \int \frac{dx}{\dot{x}^3}\left\{dx\left[\int \left(\frac{\partial h}{\partial \ddot{x}}\, dx\right) dx\right]\right\}\partial_t = F_3(x, I_1, I_2)\partial_t, \end{aligned} \tag{2.12}$$

where F_2 and F_3 are the formal expressions consequent on the conversion of (2.2) for $\ddot{x}$ and (2.5) for $\dot{x}$ in terms of I_1 and I_2.

Evidently Γ_2 and Γ_3 are generalised symmetries and not nonlocal symmetries. Since I_1 and I_2 are first integrals, it is equally evident that both $[\Gamma_1, \Gamma_2] = 0$ and $[\Gamma_1, \Gamma_3] = 0$. The Lie bracket of Γ_2 and Γ_3 is not so obvious. Since

$$\Gamma_2^{[2]} = \left(\int \frac{dt}{\dot{x}^2}\right)\partial_t + 0\partial_x - \frac{1}{\dot{x}}\partial_{\dot{x}} + 0\partial_{\ddot{x}}, \tag{2.13}$$

$$\begin{aligned}\Gamma_3^{[2]} = {} & F_3\partial_t + 0\partial_x - \frac{1}{\dot{x}}\exp\left[\int\left(\int \frac{\partial h}{\partial \ddot{x}}\,dx\right)dx\right]\partial_{\dot{x}} \\ & - \left(\int \frac{\partial h}{\partial \ddot{x}}\,dx\right)\exp\left[\int\left(\int \frac{\partial h}{\partial \ddot{x}}\,dx\right)dx\right]\partial_{\dot{x}}\partial_{\ddot{x}},\end{aligned} \tag{2.14}$$

it is evident that

$$\Gamma_2^{[2]} I_1 = 0, \qquad \Gamma_2^{[2]} I_2 = -1. \tag{2.15}$$

The calculation for Γ_3 is more delicate, but we see that

$$\Gamma_3^{[2]} I_1 = -1, \qquad \Gamma_3^{[2]} I_2 = 0. \tag{2.16}$$

As a consequence of (2.15) and (2.16) we have that

$$[\Gamma_2, \Gamma_3] = 0 \tag{2.17}$$

and so the mutual Lie brackets are all zero and the Lie algebra of the symmetries is Abelian, denoted by $3A_1$ in the Mubarakzyanov classification scheme [17–19].

Equation (2.1) is integrable in the sense of reduction to quadratures. By inspection there is just the one Lie point symmetry $\Gamma_1 = \partial_t$. Although in principle there exist two first integrals associated with Γ_1, the realisation of those integrals requires additional symmetry. In the absence of additional point symmetries the symmetries are found in what ostensibly are nonlocal symmetries, but which in reality are generalised symmetries due to the existence of the first integrals I_1 and I_2 [15].

2.2. Example

A member of the class of Eq. (2.1) is

$$\dddot{x} = -\frac{2+\beta}{\beta}x^{-1}\dot{x}\ddot{x} + f(x)\dot{x}, \quad \beta \neq 0, \infty. \tag{2.18}$$

This equation is a generalisation of the third order ordinary differential equation obtained from the reduction of the three-dimensional system

$$\dot{x} = yz, \qquad \dot{y} = zx, \qquad \dot{z} = xy, \tag{2.19}$$

an exemplar of the Rikitaki system [13], to a single third order ordinary differential equation. From (2.19) we find the third order equation

$$\dddot{x} = x^{-1}\dot{x}\ddot{x} + 2x^2\dot{x}. \tag{2.20}$$

Naturally (2.20), being a specific equation, has greater symmetry than (2.18) [2].

To treat (2.18) in the formalism of Section 1 we write

$$\frac{d\ddot{x}}{dx} = -\frac{2+\beta}{\beta}x^{-1}\ddot{x} + f(x), \tag{2.21}$$

which is a first order nonhomogeneous equation in $\ddot{x}$. Multiplication of (2.21) by its obvious integrating factor gives

$$0 = \frac{d}{dx}\left(\ddot{x}x^{(2+\beta)/\beta}\right) - x^{(2+\beta)/\beta}f(x) \tag{2.22}$$

so that

$$I_1 = \ddot{x}x^{(2+\beta)/\beta} - \int x^{(2+\beta)/\beta}f(x)\,dx. \tag{2.23}$$

We invert (2.23) to obtain

$$\ddot{x} = I_1x^{-(2+\beta)/\beta} + x^{-(2+\beta)/\beta}\int x^{(2+\beta)/\beta}f(x)\,dx, \tag{2.24}$$

from which the second first integral,

$$I_2 = \frac{1}{2}\dot{x}^2 + \frac{\beta}{2}I_1x^{-2/\beta} - \int x^{-(2+\beta)/\beta}\left(\int x^{(2+2\beta)/\beta}f(x)\,dx\right)dx, \tag{2.25}$$

follows immediately. The execution of the final quadrature to find the invariants is not, as is the case with most quadratures, expected to be possible in closed form, but the integrability of the equation has been successfully demonstrated.

3. The fourth order equation

3.1. The general equation

The fourth order form of (1.4), videlicet

$$\ddddot{x} = h(x, \dddot{x})\dot{x}, \tag{3.1}$$

is reduced to the third order equation

$$\frac{d\dddot{x}}{dx} = h(x, \dddot{x}), \tag{3.2}$$

for which the corresponding first integral is

$$I_1 = g(x, \dddot{x}). \tag{3.3}$$

Following the application of the Basic proposition we obtain

$$\dot{x}\frac{d}{dx}\left(\dot{x}\frac{d}{dx}\dot{x}\right) = f(x, I_1), \tag{3.4}$$

where f comes from the inversion of (3.3). Now we do not have an elementary first order equation, but a second order equation of somewhat more complicated structure. If we denote $\dot{x}^2$ by y and use the identity

$$\frac{1}{2}\dot{x}\frac{d^2}{dx^2}(\dot{x}^2) \equiv \dot{x}^2\frac{d^2\dot{x}}{dx^2} + \dot{x}\left(\frac{d\dot{x}}{dx}\right)^2,$$

it becomes evident that (3.4) takes the form of a generalised Emden–Fowler equation of specific index, videlicet

$$y'' = 2f(x, I_1)y^{-1/2}, \tag{3.5}$$

where the prime denotes differentiation with respect to the new independent variable x.

The Emden–Fowler equation of any index is well known to be rather sparing of integrable cases. However, there are certain integrable cases and these are readily identified as those which possess two Lie point symmetries. We indicate the calculation of these and the constraint placed on $f(x, I_1)$ for the two to exist. For more details about these readers are referred to some papers devoted to the symmetries and first integrals/invariants of the Emden–Fowler and related equations [9,11,14,16], in particular the more recent exhaustive study by Euler [4].

The determining equations for (3.5) to possess a Lie point symmetry $\xi(x, y)\partial_x + \eta(x, y)\partial_y$ are

$$\begin{aligned}
&\frac{\partial^2\xi}{\partial y^2} = 0, \qquad \frac{\partial^2\eta}{\partial y^2} = 2\frac{\partial^2\xi}{\partial x\partial y},\\
&2\frac{\partial^2\eta}{\partial x\partial y} - \frac{\partial^2\xi}{\partial x^2} - 6fy^{-1/2}\frac{\partial\xi}{\partial y} = 0,\\
&\frac{\partial^2\eta}{\partial x^2} - 4fy^{-1/2}\frac{\partial\xi}{\partial x} = 2\xi f'y^{-1/2} - \eta f y^{-3/2}.
\end{aligned} \tag{3.6}$$

From (3.6) we have

$$\xi = a(x), \qquad \eta = c(x)y \tag{3.7}$$

subject to the constraints

$$\begin{aligned}
&a'' = 2c', \qquad c'' = 0,\\
&af' + 2a'f = \frac{1}{2}cf,
\end{aligned} \tag{3.8}$$

so that there is consistency if

$$\begin{aligned}
&c = C_0 + C_1x, \qquad a = A_0 + A_1x + C_1x^2,\\
&\frac{f'}{f} = \frac{1}{2}\frac{c}{a} - 2\frac{a'}{a}.
\end{aligned} \tag{3.9}$$

However, for the presence of two symmetries f must be independent of the arbitrary constants in the coefficient functions $a(x)$ and $c(x)$. This requires that

$$f = (k_1x + k_0)^{-5/2}, \tag{3.10}$$

for which the two symmetries are

$$\begin{aligned}
&\Gamma_1 = -3(k_1x + k_0)\partial_x + y\partial_y,\\
&\Gamma_2 = (k_1x + k_0)^2\partial_x + (k_1x + k_0)y\partial_y,
\end{aligned} \tag{3.11}$$

i.e., a rescaling symmetry (Γ_1) and a projective symmetry (Γ_2).

For the purposes of the continuing discussion we use

$$\Gamma_1 = -3x\partial_x + y\partial_y, \qquad \Gamma_2 = x^2\partial_x + xy\partial_y \tag{3.12}$$

as the symmetries of

$$y'' = 2I_1 x^{-5/2} y^{-1/2}, \tag{3.13}$$

which is the form of (3.5) with two Lie point symmetries and so clearly integrable in terms of the criterion of the Lie theory. Note that we interpret the arbitrary constant in the right side of (3.13) as the value of the first integral of our original equation (3.1) as now constrained by the requirement that (3.5) be an integrable instance of the generalised Emden–Fowler equation of index $-1/2$.

We recognise in (3.12) two of the three symmetries found in $\mathrm{sl}(2, R)$ and recall [12] the relationship between the first and third elements of $\mathrm{sl}(2, R)$ in their standard representation. Under the transformation

$$X = -\frac{1}{x}, \qquad Y = \frac{y}{x}, \tag{3.14}$$

(3.4) and (3.13) become

$$\Gamma_1 = -3X\partial_X + 4Y\partial_Y, \qquad \Gamma_2 = \partial_X \tag{3.15}$$

and

$$Y'' = 2I_1 Y^{-1/2}, \tag{3.16}$$

respectively. The first and second quadratures of (3.16) give

$$\begin{aligned} I_2 &= \frac{1}{2}Y'^2 - 4I_1 Y^{1/2}, \\ I_3 &= X - \int \frac{dY}{(2I_2 + 8I_1 Y^{1/2})^{1/2}} \\ &= X + \frac{I_2}{8I_1^2}(2I_2 + 8I_1 Y^{1/2})^{1/2} - \frac{1}{48I_1^2}(2I_2 + 8I_1 Y^{1/2})^{3/2}. \end{aligned} \tag{3.17}$$

We have the three autonomous integrals of (3.1) subject to the constraint on the functional form of $h(x, \ddot{x})$ expressed in (3.13). Comparing this with (3.5) we have

$$\dddot{x} = f(x, I_1) = I_1 x^{-5/2}, \tag{3.18}$$

so that the original fourth order equation is

$$2x\ddddot{x} + 5\dot{x}\dddot{x} = 0, \tag{3.19}$$

an equation which has appeared in [4,10,11].

3.2. An elementary example

It is instructive to follow the procedure of this section for the elementary equation

$$\ddddot{x} = 0, \tag{3.20}$$

of which we know everything. We follow the formal procedure developed above. Thus

$$\frac{d\dddot{x}}{dx} = 0 \quad \Rightarrow \quad I_1 = \dddot{x} \tag{3.21}$$

so that

$$\dot{x}\frac{d}{dx}\left(\dot{x}\frac{d}{dx}\dot{x}\right) = I_1 \tag{3.22}$$

and the Emden–Fowler equation is

$$\frac{d^2y}{dx^2} = 2I_1 y^{-1/2}, \tag{3.23}$$

which is, not surprisingly, the two symmetry form (3.16) with the symmetries (3.15).

The first integral and invariant of (3.23) are (actually (3.17) in lower case)

$$I_2 = \frac{1}{2}y'^2 - 4I_1 y^{1/2},$$
$$I_3 = x + \frac{I_2}{8I_1^2}\left(2I_2 + 8I_1 y^{1/2}\right)^{1/2} - \frac{1}{48I_1^2}\left(2I_2 + 8I_1 y^{1/2}\right)^{3/2}. \tag{3.24}$$

Together with I_1 these constitute the set of three autonomous integrals of (3.20). Specifically we have, after a modicum of simplification and adjustment,

$$I_1 = \dddot{x}, \qquad I_2 = \dot{x}\dddot{x} - \frac{1}{2}\ddot{x}^2,$$
$$I_3 = x\dddot{x}^2 + \frac{1}{3}\ddot{x}^3 - \dot{x}\ddot{x}\dddot{x}, \tag{3.25}$$

which integrals can be related to the standard representation of the first integrals of (3.20) as presented in, for example, Flessas et al. [6].

It is evident from the foregoing that (3.19) and (3.20) are related through the common Emden–Fowler equation (3.16) (equally (3.23)). To make the progression of variables clearer we write (3.19) and (3.20) as

$$2x\ddddot{x} + 5\dot{x}\dddot{x} = 0, \qquad X'''' = 0, \tag{3.26}$$

where overdot denotes differentiation with respect to t and prime with respect to T. The connection between the variables in the two equations is established through (3.14). We have

$$X = -\frac{1}{x}, \qquad X'^2 = \frac{\dot{x}^2}{x}, \tag{3.27}$$

from which it follows that

$$T = \int \frac{dt}{x^{3/2}}. \tag{3.28}$$

Remark. We remark that this linearisation of (3.26) was also established independently in our recent paper [5], where we present the full classification of third order ordinary differential equations linearisable to $X''' = 0$ under the Sundman transformation

$$X(T) = F(x,t), \qquad dT = G(x,t)\,dt.$$

The second of (3.26) has the eight Lie point symmetries

$$\begin{aligned}
&\Gamma_1 = \partial_X, && \Gamma_5 = X\partial_X,\\
&\Gamma_2 = T\partial_T, && \Gamma_6 = \partial_T,\\
&\Gamma_3 = \frac{1}{2}T^2\partial_X, && \Gamma_7 = T\partial_T + \frac{3}{2}X\partial_X,\\
&\Gamma_4 = \frac{1}{6}T^3\partial_X, && \Gamma_8 = T^2\partial_T + 3TX\partial_X,
\end{aligned} \tag{3.29}$$

and the fundamental linear integrals [7]

$$\begin{aligned}
&J_1 = \frac{1}{6}T^3X''' - \frac{1}{2}T^2X'' + TX' - X, \qquad J_2 = \frac{1}{2}T^2X''' - TX'' + X',\\
&J_3 = TX''' - X'', \qquad J_4 = X''',
\end{aligned} \tag{3.30}$$

from which one may construct the three first integrals

$$\begin{aligned}
&I_1 = J_4 = X''', \qquad I_2 = J_2J_4 - \frac{1}{2}J_3^2 = X'X''' - \frac{1}{2}X''^2,\\
&I_3 = -J_1J_4^2 + \frac{1}{3}J_3^3 - J_2J_3J_4.
\end{aligned} \tag{3.31}$$

The coefficient functions of two symmetries $\gamma = \tau\partial_t + \sigma\partial_x$ and $\Gamma = \xi\partial_T + \eta\partial_X$ of the two equations in (3.26) are related according to

$$\sigma = \frac{\eta}{X^2}, \qquad 2\dot{x}(\dot{\sigma} - \dot{x}\dot{\tau}) = \frac{1}{X^2}\big[\eta Y - 2(\eta' - X'\xi')XX'\big], \tag{3.32}$$

so that the symmetries corresponding to Γ_1 through Γ_8 are

$$\begin{aligned}
\gamma_1 &= \left(\frac{3}{2}\int x\,dt\right)\partial_t + x^2\partial_x,\\
\gamma_2 &= \left(\frac{3}{2}\int x\left(\int\frac{dt}{x^{3/2}}\right)dt\right)\partial_t + \left(x^2\int\frac{dt}{x^{3/2}}\right)\partial_x,\\
\gamma_3 &= \left(\frac{3}{4}\int x\left(\int\frac{dt}{x^{3/2}}\right)^2 dt\right)\partial_t + \frac{1}{2}\left(x^2\left(\int\frac{dt}{x^{3/2}}\right)^2\right)\partial_x,\\
\gamma_4 &= \left(\frac{1}{4}\int x\left(\int\frac{dt}{x^{3/2}}\right)^3 dt\right)\partial_t + \frac{1}{6}\left(x^2\left(\int\frac{dt}{x^{3/2}}\right)^3\right)\partial_x,\\
\gamma_5 &= \frac{1}{2}t\partial_t - x\partial_x, \qquad \gamma_6 = \partial_t, \qquad \gamma_7 = \frac{5}{2}t\partial_t - \frac{3}{2}x\partial_x,\\
\gamma_8 &= -\frac{1}{2}\left(\int\left(\int\frac{dt}{x^{3/2}}\right)dt\right)\partial_t - 3x\left(\int\frac{dt}{x^{3/2}}\right)\partial_x.
\end{aligned} \tag{3.33}$$

We note that γ_1, γ_5, γ_6 and γ_7 are known from other studies [10]. It may not be surprising that the others have not previously been identified.

It is quite evident that the simple technique used to obtain the three-dimensional Abelian algebra of the third order equation in Section 2 is not likely to be successful in this case since only γ_1 of the elements $\gamma_1, \dots, \gamma_4$ of the $4A_1$ subalgebra has much hope of being determined even a postiori.

4. The fifth order equation

Under the same process as outlined above we reduce

$$x^{(5)} = h(x, x^{(4)})\dot{x} \tag{4.1}$$

to the third order equation

$$\dot{x}\frac{d}{dx}\left(\dot{x}\frac{d}{dx}\left(\dot{x}\frac{d}{dx}\dot{x}\right)\right) = f_2(x, I_{12}), \tag{4.2}$$

where I_{12} is the value of the first integral obtained from (4.1) and f_2 is its inversion to give $x^{(4)}$.

We have written the left side of (4.2) in this specific way to highlight the presence of the terms which led to the Emden–Fowler equation in our considerations of the fourth order equation. In fact with $y = \dot{x}^2$, as above, we may write (4.2) as the system of equations

$$\frac{d^2y}{dx^2} = 2f_1(x, I_{11})y^{-1/2}, \tag{4.3}$$

$$\frac{df_1}{dx} = f_2(x, I_{12})y^{-1/2}, \tag{4.4}$$

in which I_{11} is included in f_1 to indicate that there should be an arbitrary constant of integration, i.e., the value of a first integral, in it.

Equation (4.3) is the same Emden–Fowler equation of Section 3 and on the insistence that it possess two Lie point symmetries and so be integrable in the sense of Lie we have

$$f_1 = kx^{-5/2} \tag{4.5}$$

as before and identify k with I_{11}. The admissible form of (4.1) follows from the solution of (4.4).

An alternative procedure is to look again at (4.2) in the form

$$x^{(4)} = f_2(x, I_{12}), \tag{4.6}$$

which admits the integrating factor $\dot{x}$ so that one obtains

$$\dot{x}\dddot{x} - \frac{1}{2}\ddot{x}^2 = F_2(x, I_{12}, I_2), \tag{4.7}$$

where I_2 is the next constant of integration. We change independent variable from t to x. Then (4.7) becomes

$$\dot{x}^{5/2}\frac{d^2}{dx^2}(2\dot{x}^{3/2}) = F_2(x, I_{12}, I_2) \tag{4.8}$$

and on putting $y = \dot{x}^{3/2}$ we obtain a generalised Emden–Fowler equation of order $-5/3$, videlicet

$$y'' = F_2(x, I_{12}, I_2) y^{-5/3}. \tag{4.9}$$

This equation admits two Lie point symmetries if

$$F_2(x, I_{12}, I_2) = k(x + c)^{-10/3}. \tag{4.10}$$

Under the transformation

$$X = -\frac{1}{x + c}, \qquad Y = \frac{y}{x + c}, \tag{4.11}$$

(4.9) (with (4.10) taken into account) becomes

$$Y'' = kY^{-5/3}, \tag{4.12}$$

which is reduced to the quadrature

$$I_4 = X - \int \frac{dY}{(I_3 - (4/3)kY^{-2/3})^{1/2}}. \tag{4.13}$$

5. Discussion and conclusion

There is no doubt that the success of this method relies heavily on the ability to invert I_1 to solve it for $\ddot{x}$, not to mention the lengthier task for higher order equations. The choice of structure enables a formal commencement to the dual process of integration and reduction. The practical completion of that process requires some friendliness in the structure of the integrals so that, in the case of the original third order ordinary differential equation, $\ddot{x}$ is painlessly eliminated to give a function of x and I_1. In the case of equations which arise in modelling situations the dependent variable may well appear in an complicated functional form. However, the derivatives tend to appear in integral multiplicative powers as in the result of application of the chain rule for differentiation. So the type of model we desire for this scheme is one in which there is a pairing of certain derivatives and an absence of other derivatives from that combination. Thus for a second order ordinary differential equation we have $(\ddot{x}, x)$; for a third order ordinary differential equation we have $(\dddot{x}, \dot{x})$ with $(x, \ddot{x})$ in arbitrary combination. Here we have concentrated on the general pairings $(x^{(n)}, \dot{x})$ and $(x, x^{(n-1)})$. There may be other combinations which merit study. To return to our original remark; one must place some constraints. The most successful procedure for integration is that one which achieves the happy combination of minimal constraint and useful outcome.

Acknowledgments

P.G.L. Leach thanks Prof. N. Euler and Docent M. Euler and the Department of Mathematics, Luleå University of Technology, for their kind hospitality when this work was initiated, STINT (The Swedish Foundation for International Cooperation in Research and Higher Education) for its sponsorship, and acknowledges the continued support of the National Research Foundation of South Africa and the University of Natal.

References

[1] B. Abraham-Shrauner, A. Guo, Hidden symmetries associated with the projective group of nonlinear first order ordinary differential equations, J. Phys. A 25 (1992) 5597–5608.
[2] K. Andriopoulos, P.G.L. Leach, G.P. Flessas, Complete symmetry groups of ordinary differential equations and their integrals: some basic considerations, J. Math. Anal. Appl. 262 (2001) 256–273.
[3] W.E. Boyce, R.C. DiPrima, Elementary Differential Equations and Boundary Value Problems, Wiley, New York, 1969.
[4] N. Euler, Transformation properties of $x'' + f_1(t)x' + f_2(t)x + f_3(t)x^n = 0$, J. Nonlinear Math. Phys. 4 (1997) 310–337.
[5] N. Euler, T. Wolf, P.G.L. Leach, M. Euler, Linearisable third order ordinary differential equations and generalised Sundman transformations: The case $X''' = 0$, Acta Appl. Math. 76 (2003) 89–115, nlin.SI/0203028.
[6] G.P. Flessas, K.S. Govinder, P.G.L. Leach, Remarks on the symmetry Lie algebras of first integrals of scalar third order ordinary differential equations with maximal symmetry, Bull. Greek Math. Soc. 36 (1994) 63–79.
[7] G.P. Flessas, K.S. Govinder, P.G.L. Leach, Characterisation of the algebraic properties of first integrals of scalar ordinary differential equations of maximal symmetry, J. Math. Anal. Appl. 212 (1997) 349–374.
[8] K.S. Govinder, P.G.L. Leach, On the determination of nonlocal symmetries, J. Phys. A 28 (1995) 5349–5359.
[9] P.G.L. Leach, First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, J. Math. Phys. 26 (1985) 2510–2514.
[10] P.G.L. Leach, Analysis of an equation arising in general relativity, in: R. Vichnevetsky, J.J.H. Miller (Eds.), IMACS '91 Proceedings, Criterion, Dublin, 1991, pp. 363–364.
[11] P.G.L. Leach, R. Maartens, S.D. Maharaj, Self-similar solutions of the generalized Emden–Fowler equation, Internat. J. Nonlinear Mech. 27 (1992) 575–582.
[12] P.G.L. Leach, B. Abraham-Shrauner, K.S. Govinder, Symmetries of first integrals and their associated differential equations, J. Math. Anal. Appl. 235 (1999) 58–83.
[13] P.G.L. Leach, S. Cotsakis, G.P. Flessas, Symmetry, singularity and integrability in complex dynamics: I. The reduction problem, J. Nonlinear Math. Phys. 7 (2000) 445–479.
[14] P.G.L. Leach, R. Maartens, S.D. Maharaj, Self-similar solutions of the generalized Emden–Fowler equation, Internat. J. Nonlinear Mech. 27 (1992) 575–582.
[15] F.M. Mahomed, P.G.L. Leach, Contact symmetries of second order differential equations, J. Math. Phys. 32 (1991) 2051–2055.
[16] C.M. Mellin, F.M. Mahomed, P.G.L. Leach, Solution of generalized Emden–Fowler equations with two symmetries, Internat. J. Nonlinear Mech. 29 (1994) 529–538.
[17] G.M. Mubarakzyanov, On solvable Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 32 (1963) 114–123.
[18] G.M. Mubarakzyanov, Classification of real structures of five-dimensional Lie algebras, Izv. Vyssh. Uchebn. Zaved. Mat. 34 (1963) 99–106.
[19] G.M. Mubarakzyanov, Classification of solvable six-dimensional Lie algebras with one nilpotent base element, Izv. Vyssh. Uchebn. Zaved. Mat. 35 (1963) 104–116.